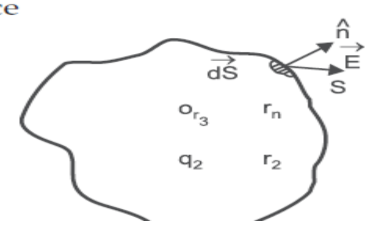
GAUSS THEOREM

ELECTRIC FLUX : The electric flux through a surface is defined as the surface integral of electric field over the surface. It is a measure of the total number of electric field lines passing normal to the surface.
It is a scalar. Its SI unit is Cm^2.

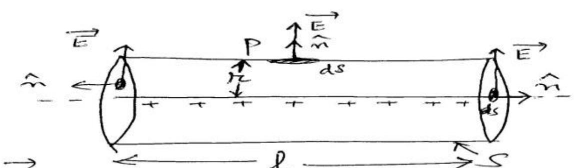
Gauss theorem in electrostatics: It states that the surface integral of electric field over a closed surface is $\frac{1}{\epsilon_0}$ times the total charge enclosed in the surface.
For surface S

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0} \text{ (Total charge inside } S\text{)}$$



Applications of Gauss Theorem

1.FIELD DUE TO INFINITELY LONG STRAIGHT LINE OF CHARGE: Consider an infinitely long thin uniformly charged wire. Let P be a point at perpendicular distance r from the wire.
As the line charge is of infinite expanse, the field lines outside the wire are radial and would be perpendicular to the wire.
Consider a cylindrical gaussian surface S of radius r and length l with its ends perpendicular to the wire as shown below in the fig.



We have;
Electric flux through the gaussian surface

$$\phi_E = \oint_S \vec{E}\cdot\vec{dS}$$
$$= \int_{\text{circular ends}} \vec{E}\cdot\vec{dS} + \int_{\text{curved surface}} \vec{E}\cdot\vec{dS}$$
$$= \int_{\text{circular ends}} E\,ds\cos 90^\circ + \int_{\text{curved surface}} E\,dS\cos 0^\circ$$
$$= 0 + \int_{\text{curved surface}} E\,dS = E\int_{\text{curved surface}} dS$$
$$= E.\,2\pi r l \quad \ldots\ldots (1)$$

By Gauss theorem; electric flux through S;

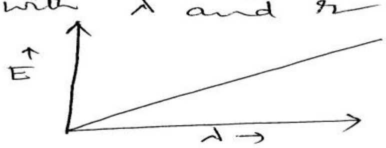
$$\phi_E = \frac{1}{\epsilon_0}(\text{Charge inside } S) = \frac{1}{\epsilon_0}(\lambda . l) \quad \ldots (2)$$

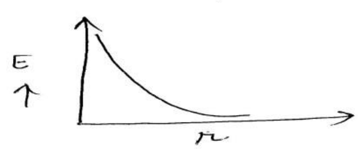
Comparing (1) and (2)

$$E.\,2\pi r l = \frac{\lambda l}{\epsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r} \Rightarrow E \propto \frac{1}{r}$$

or $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\hat{r}$

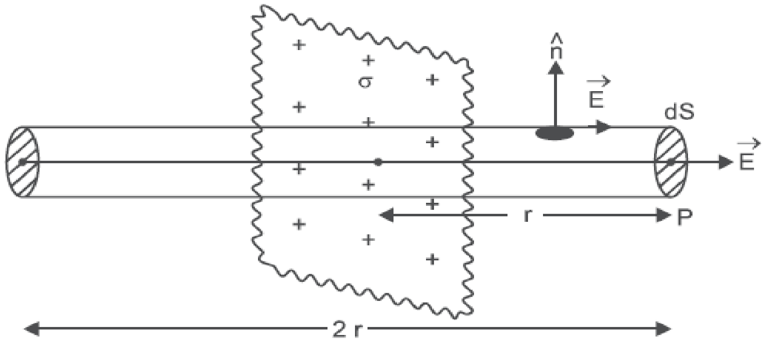
Variation of electric field due to infinite charge with λ and r





2.FIELD DUE TO INFINITELY LARGE FLAT SHEET OF CHARGE

charge: Consider a large flat sheet of charge density 'σ'. Let P be a point at a distance 'r' from the sheet.



Take a cylindrical gaussian surface 'S' with uniform area of cross-section dS and length $2r$ symmetrical about the sheet as shown.

Surface integral of the field over the surface S is given by

$$\oint_S \vec{E}\cdot\vec{dS} = \int_{\substack{\text{Curved}\\\text{surface}}} \vec{E}\cdot\vec{dS} + \int_{\substack{\text{Circular}\\\text{ends}}} \vec{E}\cdot\vec{dS}$$

$$= \int_{\substack{\text{Curved}\\\text{surface}}} EdS\cos 90^\circ + \int_{\substack{\text{Circular}\\\text{ends}}} EdS\cos 0^\circ$$

Suppo

$$= 0 + \int_{\substack{\text{Circular}\\\text{ends}}} EdS$$

$$= E\cdot\int_{\substack{\text{Circular}\\\text{ends}}} 2dS = E\cdot 2\,dS \qquad ...(1)$$

By Gauss theorem

$$\oint_S \vec{E}\cdot\vec{dS} = \frac{1}{\epsilon_0}\text{ (charge inside } S)$$

$$= \frac{1}{\epsilon_0}(\sigma\, dS) \qquad ...(2)$$

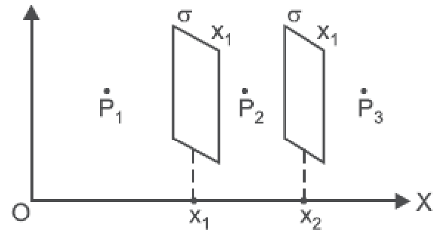
From (1) and (2)

$$E\cdot 2\,dS = \frac{1}{\epsilon_0}(\sigma\, dS)$$

$$\Rightarrow \qquad E = \frac{\sigma}{2\epsilon_0}$$

or $\qquad \vec{E} = \dfrac{\sigma}{2\epsilon_0}\hat{r}$ $\qquad$ [$\because$ The field is directed away from the positive charge]

Suppose there are two large flat sheets of charge with surface charge density σ each as shown, then



Field intensity at P_1 due to the plates X_1 and X_2

$$= \left(\frac{\sigma}{\epsilon_0} + \frac{\sigma}{\epsilon_0}\right) \text{ towards left}$$

$$= -\frac{2\sigma}{\epsilon_0}$$

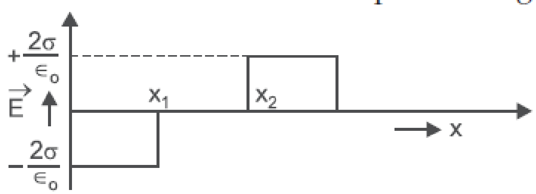
The negative sign indicates that the field is in – X-direction.

At P_2 $\quad \vec{E} = +\frac{\sigma}{\epsilon_0} \text{ (due to } X_1) - \frac{\sigma}{\epsilon_0} \text{ (due to } X_2)$

$$= \text{Zero.}$$

At P_3 $\quad \vec{E} = +\left|\frac{\sigma}{\epsilon_0} + \frac{\sigma}{\epsilon_0}\right| = \frac{2\sigma}{\epsilon_0}$ in + X-direction.

The variation of the electric field with x is represented graphically as under:



If charge densities are σ_1 and σ_2 then E_1 (at P_1)= $-(\sigma_1+\sigma_2)/2\varepsilon_0$

$E_2= (\sigma_1-\sigma_2)/2\varepsilon_0$

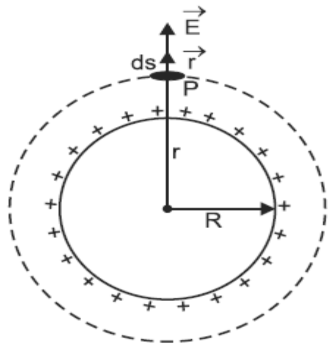
$E_1= (\sigma_1+\sigma_2)/2\varepsilon_0$

Special case: $\sigma_1=-\sigma_2=\sigma$

Then $E_1=E_3=0$ and $E_2=\sigma/\varepsilon_0$

3. FIELD DUE TO A UNIFORMLY CHARGED SPHERICAL SHELL

Consider a point *P* at a distance *r* from centre '*O*' of a spherical shell of radius *R*. Let *q* be the charge on the shell.



Take a spherical Gaussian surface *S* with centre *O* and radius *r*.
For an area *ds* of *S*, we have

$$\vec{E} \cdot \vec{ds} = Eds \cos 0° = Eds$$

$$\phi_E = \oint \vec{E} \cdot \vec{ds} = \oint_S Eds = E \cdot 4\pi r^2 \quad ...(1)$$

By Gauss Theorem,

$$\phi_E = \oint \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon_0} \text{ (charge inside } S) \quad ...(2)$$

From (1) and (2)

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} \text{ (charge inside } S)$$

(*i*) Outside ⇒ $r > R$

$$E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} q$$

$$E = \frac{1}{4\pi \epsilon_0} \frac{q}{r^2}.$$

It is same as electric field due to a point charge.

(*ii*) Inside ⇒ $r < R$

$$E \cdot 4\pi r^2 = 0 \qquad [\because q = 0]$$

$$E = 0.$$

Electric field inside a charge shell is zero.