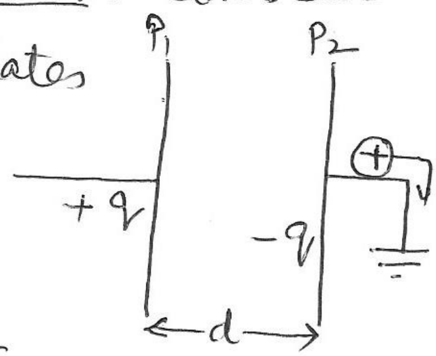
Capacitance of a parallel plate capacitor :- Consider a parallel plate capacitor with plates of area 'A' each held a distance 'd' apart. Give a charge $+q$ to P_1.



It induces a charge '$-q$' on left side and $+q$ on right side of plate P_2.

The positive charge on P_2 flows to earth and $-q$ remains due to nearness of $+q$ on P_1.

Surface charge density on $P_1 = \sigma = \frac{q}{A}$

Surface charge density on $P_2 = -\sigma = \frac{-q}{A}$.

Electric field between the plates $= E = \frac{\sigma}{\epsilon_0}$

$\therefore$ Potential difference between plates $= V = E.d = \frac{\sigma}{\epsilon_0} d$

$$\therefore \quad V = \frac{1}{\epsilon_0} \sigma . A = \frac{1}{\epsilon_0} \frac{q}{A} d$$

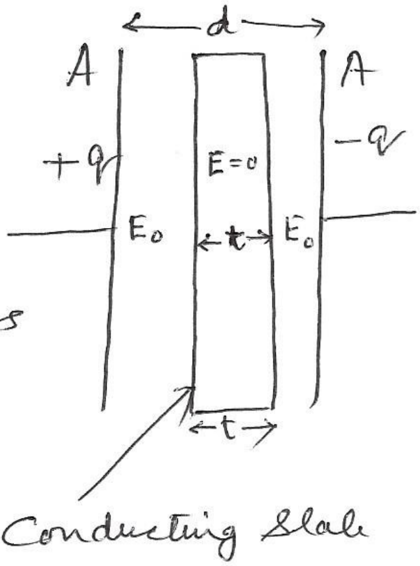
Hence $\boxed{C = \frac{q}{V} = \frac{\epsilon_0 A}{d}}$

$\Rightarrow \quad C \propto A$ and $C \propto \frac{1}{d}$.

For a parallel plate capacitor with a slab of dielectric constant K between the plates $\quad C = K \frac{\epsilon_0 A}{d}$.

Capacitance of a parallel plate capacitor with a conducting slab between the plates :

Consider a parallel plate capacitor with plates of area 'A' each held a distance 'd' apart. Suppose a conducting slab of thickness 't' is inserted between the plates. The electric field inside the electrically conducting slab becomes zero.



Hence the electric field between the plates extends only over a region $(d-t)$.

Field intensity between the plates $= E_0 = \dfrac{\sigma}{\epsilon_0}$

Potential difference between plates without slab $= V = E_0 d$

$$= \frac{\sigma}{\epsilon_0} d$$

Capacitance without slab $= C = \dfrac{q}{V} = \dfrac{\epsilon_0 A}{d}$

With slab inserted; potential difference $V' = E_0(d-t)$

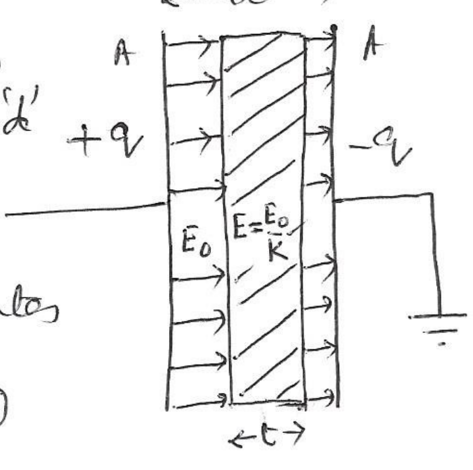
$$= \frac{\sigma}{\epsilon_0}(d-t)$$

$$= \frac{q}{\epsilon_0 A}(d-t)$$

$\therefore$ Capacitance with plate $C' = q/V' = \dfrac{\epsilon_0 A}{d-t}$

If $d = t$ i.e. the conducting slab fills entire region between plates, then $C' = \infty$.

Capacitance of a parallel plate capacitor with a dielectric slab between plates :→ Consider a parallel plate capacitor with plates of area 'A' each held a distance 'd' apart. The capacitance of the capacitor with vacuum between plates is given by



$$C = \frac{\epsilon_0 A}{d} \quad \text{—} \; ①$$

Let $E_0 \left(= \frac{\sigma}{\epsilon_0}\right)$ be the electric field intensity between the plates.

Suppose a dielectric slab of thickness 't' is inserted between the plates. If K is dielectric constant of the material of the slab; the polarisation of the slab due to electric field reduces the field intensity between the plates to $E \left(= \frac{E_0}{K}\right)$. Hence a field of intensity E_0 exists in a region $(d-t)$ and E in a region of length 't'.

Potential difference between the plates with slab becomes

$$V' = E_0(d-t) + E.t = E_0(d-t) + \frac{E_0}{K}t$$

$$= E_0\left[(d-t) + \frac{t}{K}\right] = \frac{\sigma}{\epsilon_0}\left[(d-t) + \frac{t}{K}\right]$$

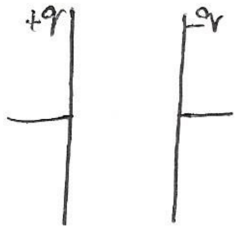
$$= \frac{1}{\epsilon_0}\left(\frac{q}{A}\right)\left[(d-t) + \frac{t}{K}\right]$$

$$\Rightarrow C' = \frac{q}{V'} = \frac{\epsilon_0 A}{(d-t) + \frac{t}{K}} = \frac{K\epsilon_0 A}{K(d-t) + t}$$

If $t = d$; then $C' = \frac{K\epsilon_0 A}{d} = KC$.

The capacitance becomes K times.

Energy stored in a parallel plate capacitor



Charging a capacitor requires transfer of electric charge from one plate to the other. Let C be the capacitance of the capacitor to be given a charge 'q' so as to increase its potential to V volt.

Let q' denote ~~the~~ an intermediate value of charge between 0 and 'q' and V' the corresponding potential. Then $V' = \frac{q'}{C}$

Suppose a small additional charge dq' increases the potential by dV' and a work dW is done in the process.

We have $dW = dq' \cdot V' = \left(\frac{q'}{C}\right) dq' = \frac{1}{C} q' dq'$

Work done to increase the charge from 0 to q is given by

$$W = \int dW = \frac{1}{C}\int_0^q q'\,dq' = \frac{1}{C}\left|\frac{q'^2}{2}\right|_0^q$$

$$= \frac{q^2}{2C} = \frac{(CV)^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}(CV).V = \frac{1}{2}qV$$

This work is stored in the capacitor as energy U.

$$\therefore U = \frac{q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}q.V$$

The energy is in the electric field between the plates.

Energy density u = Energy per unit volume $= \frac{U}{\text{Volume}} = \frac{U}{A.d}$

$$\therefore u = \frac{1}{2}CV^2 / A.d = \frac{1}{2}\frac{\epsilon_0 A}{d}(E.d)^2 \cdot \frac{1}{Ad} = \frac{1}{2}\epsilon_0 E^2$$

Note:- In addition to above; derive expressions for effective capacitance of series and parallel combination of capacitors.

Sharing of charges between capacitors :→ Suppose two capacitors with capacitance C_1 and C_2 charged to potentials V_1 and V_2 are interconnected. They share their charges till they attain a constant final potential V.

We have charge on $C_1 = q_1 = C_1 V_1$

" " $C_2 = q_2 = C_2 V_2$

Total charge $q = q_1 + q_2 = C_1V_1 + C_2V_2$

Capacitance $C = C_1 + C_2$

$\therefore$ Final ~~capacitan~~ potential $V = \dfrac{q}{C} = \dfrac{C_1V_1 + C_2V_2}{C_1 + C_2}$

This potential 'V' is known as common potential.

Energy loss on sharing charges :→ Whenever two capacitors share charges, there is always a loss of energy. In the above example; Initial energy U_i in the capacitors $= \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2$

Final energy $= U_f = \frac{1}{2}(C_1 + C_2)V^2 = \frac{1}{2}(C_1+C_2)\left(\dfrac{C_1V_1 + C_2V_2}{C_1 + C_2}\right)^2$

$$= \frac{(C_1V_1 + C_2V_2)^2}{2(C_1 + C_2)}$$

Energy loss $= \Delta U = U_i - U_f = \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2 - \dfrac{(C_1V_1 + C_2V_2)^2}{2(C_1+C_2)}$

or $\Delta U = \dfrac{C_1V_1^2(C_1+C_2) + C_2V_2^2(C_1+C_2) - (C_1V_1 + C_2V_2)^2}{2(C_1+C_2)}$

$$= \frac{C_1^2V_1^2 + C_1C_2V_1^2 + C_1C_2V_2^2 + C_2^2V_2^2 - (C_1V_1 + C_2V_2)^2}{2(C_1+C_2)}$$

$$= \frac{C_1C_2[V_1^2 + V_2^2 - 2V_1V_2]}{2(C_1+C_2)} = \frac{C_1C_2(V_1 - V_2)^2}{2(C_1+C_2)}$$

As $(V_1 - V_2)^2 \geq 0$; $\Delta U \geq 0$. Hence there always is an energy loss on sharing charges.

Important points to be remembered :→

(a) For a charged capacitor disconnected from the battery; on introducing a slab between plates

(i) q remains constant (ii) C increases (iii) V decreases

(iv) energy decreases.

(b) For a charged capacitor connected to the battery

(i) V remains constant (ii) q increases

(iii) energy increases.

(c) Force between plates of capacitor

$$F = \frac{1}{2} \frac{Q^2}{\epsilon_0 A}$$